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Refraction and shielding of sound from a source in a jet

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In typical jet-noise measurements one is almost always interested in the pressure in the far field as $kR \to \infty$. The purpose of this paper is to show that this limit is singular in the sense of matched asymptotic expansions. The inner solution ($k \to \infty$, R fixed) is given by *geometric acoustics*, whereas the outer is given by the so-called *acoustic-shielding* solution (k fixed, $R \to \infty$). A suitable composite solution is also constructed.

Now jet-noise measurements are always made at fixed and finite values of R (typically 50 jet diameters) and from these measurements one would like to infer the value of pR at $R = \infty$, where p is the acoustic pressure. One interesting and somewhat unexpected result of this paper is that the value of pR at infinity cannot be inferred from these measurements *above* a certain frequency. In other words, in order to obtain accurate estimates of pR at infinity for higher and higher frequencies, one has to be further and further away from the jet!

1. Introduction

A re-examination of the Lighthill theory of jet noise and some carefully measured data, especially for hot jets, indicate that the common interpretation of Lighthill's (1952) exact results is somewhat inaccurate (Lush 1972; Hoch *et al.* 1973). Mani, in a series of papers (1972, 1974, 1975*a*, *b*), identified the major source of discrepancy between experiment and the above theory as *acoustic/mean-flow interaction*, an effect that is missed by the usual application of Lighthill's work. Of course, a number of other authors, Ribner (1962), Csanady (1966), Pao (1973) and Gottlieb (1960), have discussed the implications of a shrouding mean flow, but perhaps not quite to the same depth as Mani did. Ribner's pioneering work in this area is especially important since it explained the presence of a relatively quiet zone in the forward angles at high frequencies.

It is now clear that acoustic/mean-flow interaction plays an extremely important part in jet-noise theories. The only real question that remains is how to include this effect in a perfectly satisfactory and tractable way. One approach, which is certainly not free from objections,† is to follow Lilley (1972); the other is to disentangle the mean-flow effects from Lighthill's equation. The latter requires a great deal of ingenuity but has essentially been done by Ffowcs

† Some of these objections were discussed by Ffowcs Williams in a recent A.I.A.A. lecture series on jet noise. The author (1975) and, more recently, Mani (1975*a*, *b*) have attempted to minimize some of these objections.

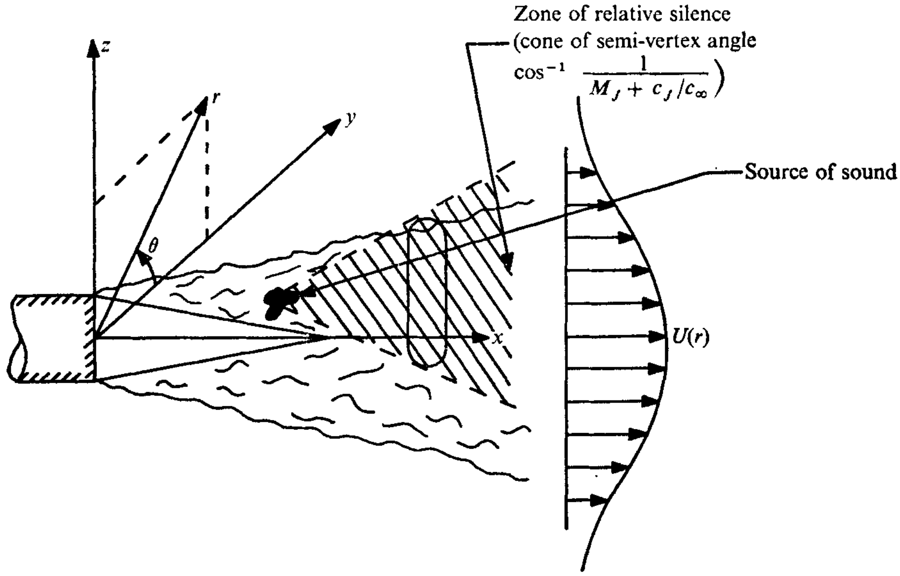


FIGURE 1. Geometry of the problem.

Williams (1974). In the present discussion of mean-flow effects we prefer to start from the Lilley formulation.

In particular, we shall focus our discussion on the sound pressure level in the forward angles (i.e. in the 'zone of relative silence')† at high frequencies. One interesting consequence of *all* the previous work on acoustic/mean-flow interaction (whether for slug or continuously sheared mean profiles) is that there appear to be *no* real acoustic rays in the zone of relative silence (figure 1). The problem that we propose to solve is how to reconcile the geometric-acoustics picture (i.e. the existence of rays everywhere) with the previous works on acoustic shielding.

At the root of the difficulty is the limit $kR \rightarrow \infty$, almost always invoked in jet-noise acoustics. Here $k = \omega/c_\infty$ (ω = circular frequency and c_∞ = speed of sound at infinity) and R is the distance from the jet. In the usual studies of mean-flow effects the above limit really implies k fixed, $R \rightarrow \infty$, whereas in geometric acoustics $k \rightarrow \infty$ with R fixed. These two limits are not interchangeable, hence $kR \rightarrow \infty$ is *singular* in the sense of matched asymptotic expansions. The net upshot of this remark is that neither geometric acoustics (an inner solution) nor the usual acoustic-shielding solution (an outer solution) can describe the exact solution uniformly as $kR \rightarrow \infty$. The purpose of the present paper is to establish the respective regions of validity of the inner and outer solutions and to construct a uniformly valid composite picture.

† Sometimes also called 'zone of silence' even though this zone is not completely silent.

2. Preliminary remarks

One objective in the study of jet noise is a solution of Lilley's equation which describes approximately the generation and propagation of sound in and through a jet. This equation is

$$L(p; U, x) = \frac{1}{c^2} D_U^3 p - D_U \Delta p - \frac{d}{dr}(\log c^2) D_U \frac{\partial p}{\partial r} + 2\frac{dU}{dr}\frac{\partial^2 p}{\partial x \partial r} = \mathscr{S}(\mathbf{x}, t), \quad (1a)$$

where $$\mathscr{S}(\mathbf{x}, t) = \rho D_U \nabla.\nabla.(\mathbf{u}'\mathbf{u}' - \overline{\mathbf{u}'\mathbf{u}'}) - 2\rho \frac{dU}{dr}\frac{\partial}{\partial x}\nabla.(u_r'\mathbf{u}' - \overline{u_r'\mathbf{u}'}), \quad (1b)$$

and in order to solve it (at least formally) it is sufficient to construct a Green's function G such that

$$L(G; U, x) = e^{-i\omega t}\delta(r - r_0)\delta(\theta)\delta(x)/r, \quad r_0 = \text{constant} \geqslant 0. \quad (2a)$$

The argument space of G is given explicitly by

$$G = G(r, \theta, x; r_0|t, \omega). \quad (2b)$$

In the above equations D_U and Δ denote a convective derivative and the Laplacian:

$$D_U = \partial/\partial t + U\partial/\partial x \quad (3a)$$

and $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \quad (3b)$$

where $U = U(r)$, $c = c(r)$ and $\rho = \rho(r)$ are the undisturbed (i.e. mean) axial component of the jet velocity, speed of sound and density and $\mathbf{x} = (r, \theta, x)$ is a cylindrical polar co-ordinate system with x pointing along the axis of the jet (figure 1). Both r_0 and ω are constants and δ denotes the delta function.

According to (1a), the pressure fluctuations p obey a wave equation which is driven by a 'known' fluctuating source distribution $\mathscr{S}(\mathbf{x}, t)$ (t is time). In the equation for the noise source, $\mathbf{u}'$ is the fluctuating turbulent velocity (u_r' is its radial component) and an overbar represents a usual statistical average. The first term of $\mathscr{S}$ is generally called *self-noise* and the second, *shear noise*. They both are quadratic in the velocity fluctuations.

Under the assumptions that G is finite on the axis $r = 0$ and represents outgoing waves at infinity, the solution for the pressure can be written as

$$p(\mathbf{x}, t) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} \mathscr{S}_0 G(r, \theta - \theta_0, x - x_0; r_0|t, \omega)\, dv_0, \quad (4a)$$

where $dv_0 = r_0\, dr_0\, d\theta_0\, dx_0$ and

$$\mathscr{S}_0 = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{i\omega t}\mathscr{S}(\mathbf{x}_0, t)\, dt. \quad (4b)$$

This representation of the pressure is valid as $t \to \infty$ since all initial conditions associated with (1a) have been ignored.

One important feature of jet noise, as pointed out by Lighthill (1952), is source convection. This implies that the Lighthill stress tensor $\mathbf{u}'\mathbf{u}'$ can be written roughly as

$$\mathbf{T} \equiv \mathbf{u}'\mathbf{u}' \simeq \mathscr{T}(\mathbf{x} - U_c t, t), \quad (5)$$

where $\mathscr{T}(\mathbf{x}, t)$ denotes the 'spatial and temporal characteristics of a stationary eddy' (usually isotropic turbulence; see Proudman 1952; Ribner 1969) and $\mathbf{U}_c = \text{constant} = (0,\ 0,\ U_c)$ is a representative source convection velocity. Although (4*a*, *b*) are valid for arbitrary $\mathscr{S}(\mathbf{x}, t)$, it is worthwhile to introduce explicitly convection effects (5) into $\mathscr{S}(\mathbf{x}, t)$, and subsequently into (1*a*). After applying the Galilean transformation $\mathbf{x}' = \mathbf{x} - \mathbf{U}_c t$ to the resultant equation, we find that the form of (1*a*) remains invariant (i.e. Galilean invariance) and the following change of variables occurs:

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} - \mathbf{U}_c t, \quad U \to V = U - U_c. \tag{6a,b}$$

$\mathscr{S}$ may then be calculated from $\mathscr{T}(\mathbf{x}', t)$.

Under the assumption that the sources are reasonably compact, most of the characteristics of p are contained in G and its derivatives, so that for the purposes of this paper it is sufficient to examine the properties of the Green's function alone. Furthermore, G will be studied under the additional constraint that $\omega a/c_\infty \gg 1$; this study should indicate the behaviour of the high frequency content of the mean-square acoustic pressure. Here a is the length scale associated with the radial gradients of the mean flow (i.e. a is essentially a shear-layer thickness) and c_∞ is the speed of sound at infinity. We remark, however, that under certain conditions (e.g. high jet velocity) non-compactness effects can be very important, as first pointed out by Ribner (1959). This is because source convection increases the time delays (at forward angles) across the source region by the Doppler factor $(1 - M_c \cos\Theta)^{-1}$.

3. Outer solution: acoustic-shielding results

In view of what has just been said about compactness and source convection, it is sufficient for the *purposes* of this paper to solve

$$L(G;\, V, x') = e^{-i\omega t}\,\delta(x')\,\delta(r)\,\delta(\theta)/r, \tag{7a}$$

$$V = U - U_c, \tag{7b}$$

where G also satisfies the radiation condition at infinity. Note that in (7*a*) the r location of the source has been set to zero; this simplification again will not influence significantly the qualitative findings of this paper (see also Mani 1975*a*, *b*).

Under the assumption† that $dU/dr < 0$ and $dc/dr < 0$, it was shown by the author (1976) that

$$|G| = \frac{1}{4\pi c_\infty kR}\frac{c_J}{c_\infty}\frac{\exp\left(-k\int_0^{r_\sigma} f\,dr\right)}{(1 - M_J\cos\Theta)^2}, \quad 0 < \Theta < \Theta_c, \tag{8a}$$

where c_J and M_J are the maximum values of the jet speed of sound and Mach number, occurring, of course, at $r = 0$ (i.e. $M_J = U(0)/c_\infty$, $c_J = c(0)$), c_∞ = constant is the speed of sound at infinity and $k = \omega/c_\infty$. Equation (8*a*) is valid as $kR \to \infty$, where R is the distance between the observer and the position of the source at the time of emission and Θ is the angle between this vector and the axis

† This assumption is satisfied for a hot round jet.

of the jet. Usually R and Θ are interpreted as the distance from the jet and the angle with respect to the jet axis. The function f, a measure of the *shielding* (i.e. the exponential decrease in the amplitude $|G|$), is given by

$$f(r) = \frac{[\cos^2\Theta - (1-M\cos\Theta)^2\,(c/c_\infty)^{-2}]^{\frac{1}{2}}}{1-M_c\cos\Theta}, \tag{8b}$$

where $M = M(r) = U(r)/c_\infty$ and $M_c = U_c/c_\infty$. For a given Θ in the range $(0, \Theta_c)$, $f(r)$ vanishes at a unique point called r_σ, which defines the upper limit of integration in (8 a).

The angle Θ_c defines the so-called zone of relative silence ($0 < \Theta < \Theta_c$) and is given by

$$\Theta_c = \cos^{-1}\frac{1}{c_J/c_\infty + M_J}. \tag{9}$$

As seen from (8 a), the sound field is *not* identically zero in this zone of silence, but merely decays exponentially as the source frequency is increased indefinitely holding the observation angle and velocity and temperature profiles fixed.

This zone of silence is also called the *refraction valley* in a number of papers by the University of Toronto group (and perhaps others). In fact, it was Ribner and some of his students who first measured experimentally the sound field in the refraction valley of a harmonic point source embedded in a jet (see, for example, Atvars *et al.* 1966). We prefer *not* to use this terminology because, in our view, refraction is generally associated with geometrical acoustics (i.e. refraction is essentially ray bending) and (8 a) has nothing to do with these geometrical concepts. Furthermore, what we call shielding also has a profound effect on the power radiated by a source embedded in a mean, shrouding flow; see Mani (1972). Thus shielding implies considerably more than just a redirection of sound. In fact, very loosely, shielding is that physical effect that converts a *non-wavelike* disturbance in the vicinity of the source into a *wavelike* disturbance very far from the source.

For $\Theta > \Theta_c$, the amplitude of the Green's function is still given by (8 a) with $r_\sigma \equiv 0$.

Let us recall that the previous results, when extended to suitable quadrupoles, explain many of the features of jet noise over a *wide* range in frequency (Balsa 1976). A similar conclusion, though for a slug-flow model, was also reached by Mani (1975 a, b).

4. Singular nature of problem for $kR \gg 1$

Although the acoustic-shielding theory described in the previous section is quite relevant to jet noise, it has one 'peculiar' characteristic, namely the exponential dependence of the pressure on frequency in the zone of relative silence. This dependence has been partially and independently confirmed by Schubert (1972) over a limited frequency range.

Two additional and important factors will determine the actual amount of shielding in the zone of silence. These are (i) turbulent scattering and (ii) the precise radial location of the sound-producing eddy. It is clear on an intuitive basis that turbulent scattering, since it is diffusive in nature, will smear out sharp gradients and thus will limit the smallness of the pressure in the zone of silence. We shall not say any more about turbulent scattering in this paper. Second, the author has shown, in work as yet unpublished, that acoustic shielding in the zone of silence can be somewhat *reduced* by placing the quadrupoles at the nozzle lip. The practical significance of the last remark is that we can now successfully predict jet noise up to reasonably high source Strouhal numbers. However, even in the off-axis case (i.e. quadrupoles at nozzle lip), the dependence of the pressure on frequency is exponential and consequently the theory will probably disagree with experiment at high enough frequencies, especially at angles near the jet axis.

Those readers who are familiar with the kinematic or geometric theories of wave propagation may find it strange that in the high frequency solution of the previous section the leading-order term (and that is the only term we consider) is exponentially small; see (8 *a*). On the basis of some very general theorems for symmetric hyperbolic systems (see, for example, Lewis 1965), we *expect* the acoustic pressure to behave as

$$p \sim \sum_{n=0}^{\infty} A_n k^{-\beta_n} + \text{transcendentally small terms in } k,$$

where $\beta_0 = 0 < \beta_1 < \beta_2, \ldots$, and the $|A_n|$ are *independent* of k.

The coefficients A_n are determined by the so-called transport equations along the rays and A_0 is called the *geometric-acoustics* solution. Since the A_n satisfy first-order linear equations along the rays and do not vanish at the source, they cannot vanish along the rays (except perhaps when certain singularities are encountered; no singularities can exist for $\Theta < \frac{1}{2}\pi$ when $dU/dr < 0$ and $dc/dr < 0$). Furthermore, from energy arguments, one would expect $A_0 \sim R^{-1}$ as $R \to \infty$. While these expectations are correct for points outside the zone of silence, they are incorrect for those within it. Thus for $k \to \infty$ with R fixed the solution (essentially A_0) is quite different from that for k fixed, $R \to \infty$ [essentially (8 *a*)]. The last remark implies that the limit $kR \to \infty$, usually invoked in jet noise, is a singular one in the sense of matched asymptotic expansions. In summary, the limit $kR \to \infty$ implies the following two regions.

(i) Inner region: $k \to \infty$, R fixed.

(ii) Outer region: k fixed, $R \to \infty$.

The *acoustic-shielding* solution of § 3 corresponds to the outer solution and cannot describe the exact solution *uniformly* as $kR \to \infty$. Thus we must construct the inner solution which is given by *geometric acoustics*. A uniformly valid solution can then be obtained by forming a suitable composite solution.

The previous remarks on the non-uniformity of the acoustic field as $kR \to \infty$ are quite reminiscent of those on the classical Stokes–Oseen approximation in viscous flow (Van Dyke 1975, p. 151).

5. Inner solution: geometric-acoustics results

The principal assumption of geometrical acoustics is that for $\omega a/c_\infty \gg 1$

$$G = e^{-i\omega t}\, e^{ik\phi}\, A(\mathbf{x}') + O(k^{-\beta}), \quad \beta > 0, \tag{10}$$

where $\phi = \phi(\mathbf{x}')$ is the phase and $A(\mathbf{x}') = |G|$ is the amplitude of the acoustic signal. Substituting (10) into Lilley's equation (7 *a*) and collecting terms involving like powers of ω yields†

$$\phi_r^2 + r^{-2}\phi_\theta^2 = g^2(r, \phi_x) \quad \text{(from terms in } \omega^3), \tag{11a}$$

where
$$g^2 = \frac{(1-N\phi_x)^2}{(c/c_\infty)^2} - \phi_x^2, \quad N = \frac{U-U_c}{c_\infty}. \tag{11b, c}$$

Equation (11 *a*) is the *eikonal* equation and its solution by the method of characteristics is standard (Courant & Hilbert 1966, p. 97). The so-called characteristics of (11 *a*) or equivalently the *rays* of (7 *a*) are given by

$$\frac{dr}{ds} = \phi_r, \quad \frac{d\theta}{ds} = \frac{\phi_\theta}{r^2}, \quad \frac{dx}{ds} = -g\frac{\partial g}{\partial \phi_x}, \tag{12a–c}$$

and along these rays we find that

$$\frac{d\phi_r}{ds} = \frac{\phi_\theta^2}{r^3} + g\frac{\partial g}{\partial r}, \tag{13a}$$

$$d\phi_\theta/ds = 0, \tag{13b}$$

$$d\phi_x/ds = 0, \tag{13c}$$

$$\frac{d\phi}{ds} = g^2 - \phi_x g\frac{\partial g}{\partial \phi_x} = \frac{1-N\phi_x}{(c/c_\infty)^2}, \tag{13d}$$

where s is a suitable parameter *along* the ray.‡ Thus (12) and (13) give seven ordinary differential equations for the seven variables r, θ, x, ϕ_r, ϕ_θ, ϕ_x and ϕ along each ray. The constancy of ϕ_x and ϕ_θ implies the existence of *two* Snell laws for Lilley's equation.

The ordinary differential equations (12) and (13) can be solved in closed form for 'outgoing' waves (there is another solution representing 'incoming' waves):

$$\phi = \phi_x x + \phi_\theta \theta + \int_0^r (g^2 - \phi_\theta^2/r^2)^{\frac{1}{2}}\, dr, \tag{14a}$$

and
$$x = \int_0^r \frac{N(1-N\phi_x) + (c/c_\infty)^2 \phi_x}{(c/c_\infty)^2 (g^2 - \phi_\theta^2/r^2)^{\frac{1}{2}}}\, dr, \tag{14b}$$

$$\theta = \phi_\theta \int_0^r \frac{dr}{r^2(g^2 - \phi_\theta^2/r^2)^{\frac{1}{2}}}, \tag{14c}$$

subject to the initial conditions that all rays pass through the source (i.e. $x = \theta = 0$ for $r = 0$) and that the phase vanishes at the source. Thus the explicit equation

† Note that for notational simplicity we shall write x for x', so that ϕ_x in (11*a*) is really $\phi_{x'}$.

‡ Note that $[\phi_r^2 + \phi_\theta^2/r^2 + g^2(\partial g/\partial \phi_x)^2]^{\frac{1}{2}}\, ds$ is the elemental arc length along the ray.

of a ray is given by (14 *b*, *c*). Clearly there is a two-parameter family of rays corresponding to permissible values of (ϕ_x, ϕ_θ). When the acoustic field is axially symmetric, as G is, $\phi_\theta \equiv 0$.

The coefficient of the ω^2 term in Lilley's equation describes the variation of the amplitude along the ray. This amplitude equation is

$$\frac{dA^2}{ds}+\left[\frac{d}{ds}\log\frac{(c/c_\infty)^2}{(1-N\phi_x)^2}+\Delta\phi-\frac{N^2}{(c/c_\infty)^2}\phi_{xr}\right]A^2=0. \tag{15}$$

We now digress for a moment to write the pertinent phase, ray and amplitude equations in a more familiar, vector form. First, after rearranging and taking the positive square root of (11 *a*), we find that

$$(1-N\phi_x)(c/c_\infty)^{-1}=|\nabla\phi|, \tag{16a}$$

and the ray equations (12) become

$$\frac{c/c_\infty}{|\nabla|}\frac{d\mathbf{X}}{ds}=\mathbf{N}+(c/c_\infty)\,\mathbf{n}, \tag{16b}$$

with $\mathbf{n}=\nabla\phi/|\nabla\phi|$, $d\mathbf{X}=(dr,\ r\,d\theta,\ dx)$ and $\mathbf{N}=(0,\ 0,\ N)$. The quantity $\mathbf{N}+(c/c_\infty)\,\mathbf{n}$ is the group or ray velocity. The corresponding vector form of (13) is

$$\frac{(c/c_\infty)}{|\nabla\phi|}\frac{d\nabla\phi}{ds}=\mathbf{e}_r\left[-\frac{\partial N}{\partial r}\phi_x-\frac{\partial(c/c_\infty)}{\partial r}|\nabla\phi|\right]=-(\nabla\mathbf{N}).\nabla\phi-|\nabla\phi|\,\nabla(c/c_\infty), \tag{17}$$

where $\mathbf{e}_r$ is a unit vector in the r direction. We may eliminate $\Delta\phi$ from the amplitude equation (15) by observing that

$$\mathbf{n}=\nabla\phi/|\nabla\phi|=\frac{c/c_\infty}{1-N\phi_x}\nabla\phi \tag{18}$$

and taking the divergence of $\mathbf{n}$. The final result is

$$\frac{c/c_\infty}{|\nabla\phi|}\frac{dA^2}{ds}+\left[\nabla.\left(\frac{c}{c_\infty}\mathbf{n}\right)-\frac{c/c_\infty}{|\nabla\phi|}\frac{d}{ds}\log\left(\frac{c}{c_\infty}|\nabla\phi|\right)\right]A^2=0, \tag{19}$$

where, as before, $d/ds=d\mathbf{X}/ds.\nabla$. The amplitude equation (19) implies that along a *fixed* ray

$$A^2\propto\frac{c}{c_\infty}\frac{|\nabla\phi|}{|\mathbf{N}+c/c_\infty\,\mathbf{n}|\mathscr{A}}=\text{RHS}, \tag{20}$$

where $\mathscr{A}$ is the ray-tube area. This ray-tube area is generated by a family of neighbouring rays and its normal is along the ray.

Equation (20) implies that $A^2\times(\text{RHS})^{-1}$ is a constant along a ray; this constant is generally called Blokhintsev's (1946) invariant, which reduces to (20) when the undisturbed static pressure is a constant as in Lilley's equation.

Before the solution for the amplitude is complete, we must compute the ray-tube area $\mathscr{A}$ and the constant of proportionality appearing in (20). The latter quantity is obtained by requiring that in the *vicinity* of the source equation (20) reduces to that for a source in a *uniform* flow of velocity $V_J=V(0)=U(0)-U_c$ and the speed of sound $c_J=c(0)$. The calculation of $\mathscr{A}$ is standard, either from geometric arguments or from the 'derived ray equations' of Hayes (1970).

The final result for the amplitude is

$$A = |G| = \frac{1}{4\pi c_\infty Rk}\frac{c_J}{c}\frac{1-N\sigma_*}{(1-N_J\sigma_*)^2}\left(\frac{r}{g_*\mathscr{A}}\right)^{\frac{1}{2}}\frac{1}{\sin\Theta}, \tag{21 a}$$

where

$$\mathscr{A} = \int_0^r \frac{dr}{(c/c_\infty)^2 g_*^3}, \quad N = \frac{V}{c_\infty} = \frac{U-U_c}{c_\infty}. \tag{21 b, c}$$

As before, R is the distance from the jet, Θ is the angle with respect to the jet axis and g_* denotes $g = [(1-N\phi_x)^2(c/c_\infty)^{-2}-\phi_x^2]^{\frac{1}{2}}$ evaluated at $\phi_x = \sigma_*$. Equation (14 *b*) and the monotonicity of U and c imply that for $\Theta < \frac{1}{2}\pi$ there is a unique ray passing through each point (r, x). This ray is identified by the unique $\phi_x = \sigma_*$.

We remark that (21 *a*) is valid for all values of R; that is, it is *not* an approximation of the solution in the far field. This is in contrast to (8 *a*), which is valid only as $R \to \infty$.

This concludes the theoretical development of the paper. The rest of this work s devoted to discussing the outer and inner solutions (8 *a*) and (21 *a*).

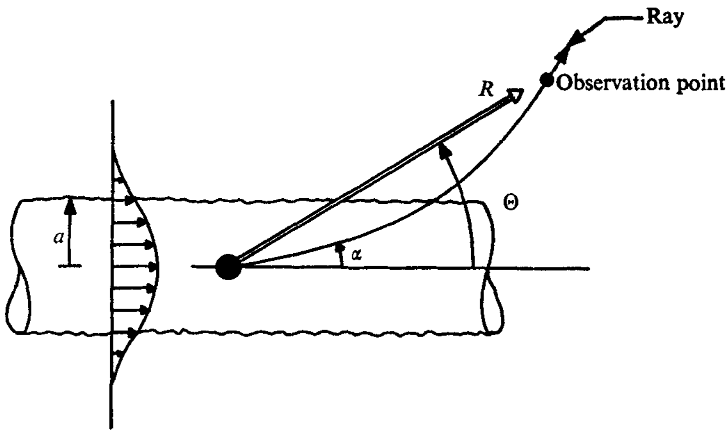
6. Discussion

The calculation of the geometric-acoustics field (21 *a*) proceeds along familiar lines. First, for each ray, that is for each value of $\phi_x = \sigma_*$ ($\phi_\theta \equiv 0$), (14 *b*) and (21 *b*) are integrated numerically to obtain the position of the ray $x = x(r)$ and the modified ray-tube area $\mathscr{A}$. Since there are no caustics for $\Theta < \frac{1}{2}\pi$, the numerical integration offers no difficulty; more specifically, $\mathscr{A}$ never vanishes.

The relationship between Θ and α is shown in figure 2. Once again, Θ is the angle between the observation vector and the jet axis, and α is the corresponding angle the ray makes (with the x axis) at the source. There are several interesting observations to be made. First, the ray that passes through $\Theta = 90°$ has an initial ray angle of $\alpha = 51°$. This ray is then *refracted* (i.e. refraction = ray bending due to mean-velocity and temperature gradients), so that far away from the jet it passes through the 90° observation point. Note that the direction of the ray is the direction along which energy propagates. This direction, in general, is *not* the same as that of the wave normal (i.e. the normal to the phase surfaces ϕ = constant). The precise relationship between the ray direction $d\mathbf{X}/ds$ and the wave normal $\nabla\phi$ is given by (16 *b*). When the flow vanishes (i.e. $\mathbf{N} = 0$) the ray is along the wave normal. However, this is a very special case that is commonly encountered only in classical acoustics.

Second, the ray that starts out horizontal remains horizontal forever since the curvature of a ray is proportional to the mean-flow gradients (which vanish at $r = 0$). Thus the point $\alpha = 0$ corresponds to $\Theta = 0$, as is shown in the figure.

Third, and most important, the curve $\alpha = \alpha(\Theta)$ exhibits a 'boundary layer' in the vicinity of $\alpha = 0$. Here the term 'boundary layer' is used in its generalized sense, as is frequently done for singular perturbations. In this particular example, the term simply denotes the region in which $d\alpha/d\Theta$ is very small. Thus the curve $\alpha = \alpha(\Theta)$ may be conveniently divided into two regions: $d\alpha/d\Theta = O(1)$ and $d\alpha/d\Theta \gg 1$. The rays that fill this boundary layer (i.e. $\alpha < 3°$) spread quite



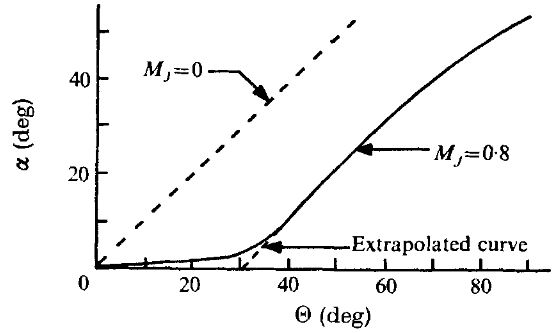


FIGURE 2. Geometric-acoustics solution. Relationship between angle to jet axis and initial ray angle. $M_c = 0$, cold jet, $\epsilon = a/R = 0{\cdot}1$.

rapidly and emerge in a cone of semi-angle of about 30°. Thus this cone is not expected to carry much acoustic energy since all of this energy must have originated from a very small part of the source.

Failure to recognize this boundary layer has led to several incorrect statements, such as "a cone ... dowstream of the source is not reached by any sound rays at all ..." (Csanady 1966) or "sound emitted from sources within a jet is expected to have an acoustic shadow inside a cone in the forward direction ..." (Morse & Ingard 1968, p. 713). (Indeed, if one extrapolates the $\alpha = \alpha(\Theta)$ curve down to the Θ axis as shown in figure 2, one arrives at the incorrect conclusion that the $\alpha = 0$ ray emerges at $\Theta \approx 30°$.) Figure 2 clearly shows that there are rays everywhere, no matter how small Θ is. Incidentally, the results in figure 2 are given for $M_c = 0$ and $\epsilon \equiv a/R = 0{\cdot}1$, where a is the jet radius (the value of r where the local velocity is about 15% of the centre-line velocity, an arbitrary definition). In all of the sample calculations the velocity profile is exponential: $M = M_J \exp(-2r^2/a^2)$.

It is precisely the presence of this boundary layer that accounts for the singular nature of the problem as $kR \to \infty$ (see §4).

In figure 3 we show the sound pressure level $SPL' \equiv 10\log_{10}(R^2|G|^2)$ for

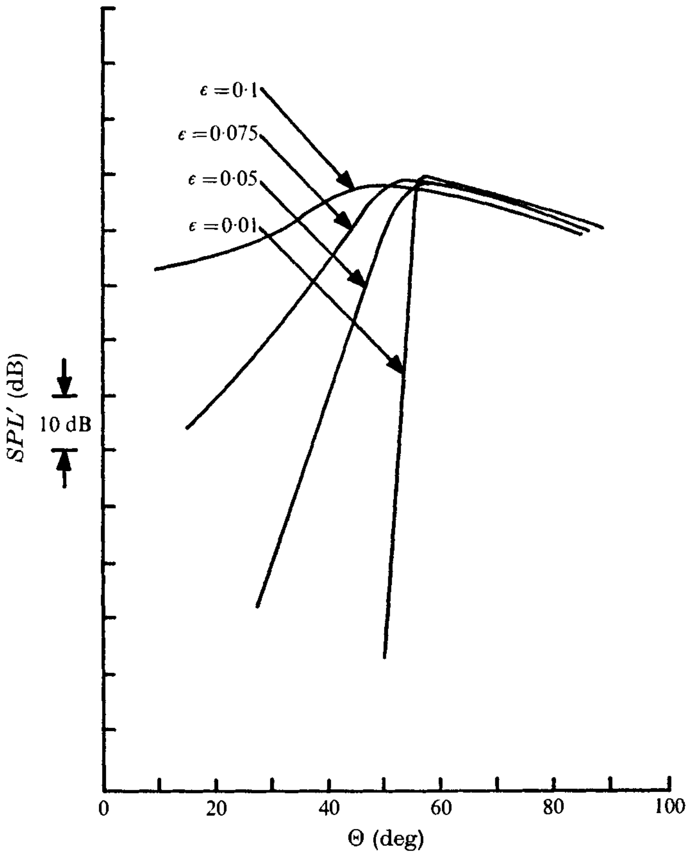


FIGURE 3. Pressure level in inner region (geometric acoustics). $M_J = 0{\cdot}8$, $M_c = 0$, cold jet.

various values of $\epsilon = a/R$. Note that the R^{-2} dependence has already been extracted from this definition of SPL' and that the calculations are made from the inner solution (21 *a*). These results show that for 'large' Θ (i.e. for Θ outside the boundary layer of figure 2) $|G|^2$ follows very nearly an R^{-2} dependence (i.e. SPL' is essentially independent of R), while for 'small' Θ (i.e. for Θ inside the boundary layer of figure 2) SPL' has substantial additional dependence on R; more precisely, the dependence is on $\epsilon = a/R$. We find that only for $\epsilon = 0$ is there a *perfect* shadow in the downstream region near the jet axis. Thus in the inner region, at high frequencies, we may write

$$p^2_{\text{inner}} \rightarrow R^{-2} f^{(i)}(\Theta, a/R; M_J, M_c, c_J/c_\infty),$$

where $f^{(i)}$ is a known function (which is actually displayed in figure 3). The dependence on $\epsilon = a/R$ enters because the mean velocity profile has the length scale a associated with it.

This decrease in the pressure level at shallow angles and for small values of ϵ is due to classical *refraction* or *ray bending.*

The next step is to compare the outer and inner solutions (8 *a*) and (21 *a*). This is done in figure 4. The acoustic-shielding results are given for three values of the

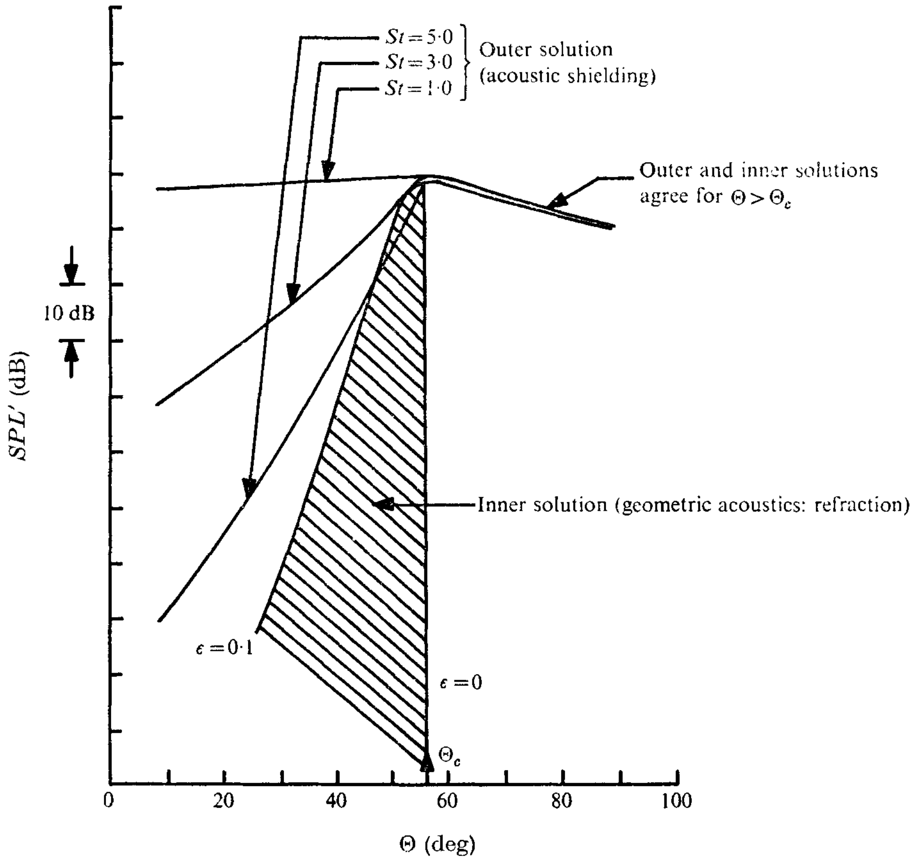


FIGURE 4. Comparison of inner and outer solutions.
$M_J = 0{\cdot}8$, $M_c = 0$, cold jet.

source Strouhal number ($St = ka/\pi M_J$) and the geometric-acoustics solution lies somewhere in the shaded region, depending on the value of ϵ. In jet-noise measurements $\epsilon = a/R$ is typically about 10^{-2}. Thus the outer solution, valid for large values of R and high frequencies, has the functional form

$$p^2_{\text{outer}} \to R^{-2} f^{(o)}(\Theta, ka; M_J, M_c, c_J/c_\infty).$$

Of course, $f^{(o)}$ is shown graphically in figure 4.

Now in jet-noise measurements one would always like to obtain the outer solution because the integral of this solution, over a suitable surface, yields the power radiated by the source. Also the outer solution is the true far-field solution since $p_{\text{outer}} \sim R^{-1}$. Since jet-noise measurements are always made at finite values of R, the measured acoustic field will always be contaminated by the inner solution. The degree of contamination is shown in figure 4. Clearly, as the frequency is increased at a fixed value of ϵ (i.e. at a fixed measuring radius for a given jet) the degree of contamination increases. For example, when $\epsilon = 0{\cdot}1$, $St = 5$ and $\Theta \approx 40°$ the inner and outer solutions are of the same order (figure 4), so that under these conditions it is *not* the acoustic far field that would be measured.

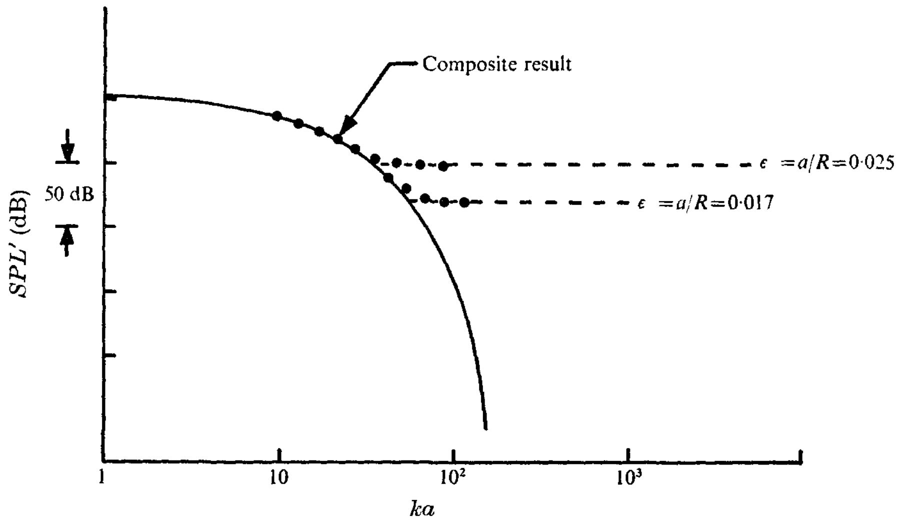


FIGURE 5. Inner, outer and composite solutions. $\Theta = 30°$, $M_J = 0{\cdot}3$, $M_c = 0$, cold jet. ———, outer solution; - - - - - -, inner solution.

Since the common asymptote of (8*a*) and (21*a*) is identically zero for $\Theta < \Theta_c$, the construction of the composite solution is extremely simple. One such solution is shown in figure 5 for $\epsilon = 0{\cdot}025$. These results indicate that, for very large values of ka, the outer solution completely breaks down. Conversely, the inner solution is not valid for moderate values of ka. Also, the inner solution limits the amount by which SPL' can fall as $ka \to \infty$ in the absence of turbulent scattering.

7. Conclusions

We have shown in this paper that the limit $kR \to \infty$ is singular in the sense of matched asymptotic expansions. One often encounters this limit while measuring the far field of noise generated by jet-like flows. The inner and outer solutions are obtained from *geometric-acoustics theory* ($k \to \infty$, R fixed) and *acoustic-shielding theories* (k fixed, $R \to \infty$) respectively.

In order to ensure that a suitable integral of the sound pressure level results in the power (or power spectrum) of the radiated sound, it is necessary to have $|G^2| \sim R^{-2}$ as $R \to \infty$. Clearly, it is only the acoustic-shielding theory that has this variation of amplitude for all Θ. However, the actual (or measured) acoustic field, especially at shallow angles, is a composite of the shielding and geometric fields (figure 5). For a certain location in the far field, say $\epsilon = a/R = 0{\cdot}025$, there is a maximum value of ka, say $ka \approx 10$, beyond which the difference between the composite and shielding results becomes larger than some error criterion, say 2 dB. Thus at this location in the far field the measured noise is no longer 'the infinitely far noise' (quite apart from the inverse-square law) when the frequency is greater than that corresponding to $ka \approx 10$.

Obviously then, the validity of a far-field location must be carefully established not only at low but also at *high* frequencies in any experimental set-up. Furthermore, it is essential to do this for jet-like flows (loudspeakers commonly used will not do) and at shallow angles.

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